Surface viscosity in nematic liquid crystals

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(Received 9 February 2009; published 8 May 2009)

We consider the effect of a localized surface viscosity on the relaxation of an imposed deformation in nematic liquid crystal cells. The simple case in which the samples are in the shape of a slab and the differential equations can be linearized is considered. The apparent inconsistence between the initial values of the time derivatives at the border evaluated by means of the bulk equation and of the boundary condition is related to the assumption that the distorting field is removed in a discontinuous manner. In this framework we shall see that the dynamical problem relevant to the relaxation of the deformation is a well posed problem. In particular, the time derivatives of the nematic director evaluated on the surface by means of the bulk differential equation and by means of the dynamical boundary condition are identical for times larger than the switching time of the deforming field. The analysis of the relaxation of the imposed deformation based on the diffusion equation, with the boundary condition containing the surface viscosity, is then valid only for times larger than the switching time of the deforming field. From this observation we conclude that the concept of localized surface viscosity is useful in the description of slow dynamics of nematic liquid crystals.

DOI: 10.1103/PhysRevE.79.051701 PACS number(s): 83.80.Xz, 61.30.Hn

I. INTRODUCTION

The dynamical description of the deformation imposed by an external field on a nematic sample, or the relaxation of an imposed deformation when the distorting field is removed, is of fundamental importance in display's technology. The mathematical description of the phenomena involves bulk and surface properties of the liquid crystal. For what concerns the bulk, the dynamical description of the liquid crystal is well understood, and it is based on its elastic constants and rotational viscosity [1]. The description of the surface properties is far from being complete. Long ago, Rapini and Papoular [2] introduced the concept of anisotropic part of the surface tension, to take into account the anisotropic interaction between a solid substrate and the adjacent nematic liquid crystal. However, if the orientation of the nematic director at the surface changes with the time, it is necessary to introduce also a surface dissipation function, related to a kind of surface viscosity, as proposed by Derzhanskii and Petrov [3]. The dynamical boundary condition based on the concept of a localized surface viscosity has been widely accepted. Several studies have been devoted to the influence of the surface viscosity on the dynamics of nematic liquid crystals [4–10]. In a recent paper Sonnet *et al.* [11] have pointed out that the introduction of the surface viscosity according to the scheme proposed in [3] has a serious problem related to the compatibility of the time derivatives of the director at the boundary, deduced by the bulk and surface dynamical equations. The aim of our paper is to show that in the simple case in which the sample of the liquid crystal is in the shape of a slab there are no problems of compatibility, in the sense that the initial time derivative of the director evaluated on the surface by means of the bulk differential equation, and by means of the dynamical boundary condition, are identical for $t>0$.

Our paper is organized as follows. In Sec. II we recall the usual continuum description of nematic liquid crystals to define in a proper manner the bulk and surface densities of elastic and viscous torques. In Sec. III the relaxation of an imposed deformation is considered and the characteristics relaxation times and wave vectors of the deformation evaluated. In the same section, the compatibility of the initial time derivatives of the director on the limiting surfaces evaluated by means of the bulk and boundary condition equation is discussed. In Sec. IV it is shown that the eigenfunctions of the dynamical problem described by the diffusion equations for the bulk with the boundary conditions involving the surface viscosity of the form a basis of Riesz [12]. From this result it follows that all initial deformation imposed by an external field can be decomposed in series of the eigenfunction of the problem.

II. CONTINUUM DESCRIPTION OF NEMATIC LIQUID CRYSTALS

We consider, first, a sample in the shape of a slab of thickness d in the static case. The Cartesian reference frame used for the description has the z axis perpendicular to the limiting surfaces located at $z=\pm d/2$. The nematic deformation is assumed to have only splay and bend contributions and contained in a plane that we call (x,z). The angle formed by the nematic director with the z axis, the so-called tilt angle, is indicated by ϕ. In the one-dimensional problem under consideration $\phi=\phi(z)$. The actual nematic distortion is the one minimizing the total energy per unit surface F given by

$$F[\phi(z)]=\int_{-d/2}^{d/2} f(\phi,d\phi/dz;z)dz+g_1(\phi_1)+g_2(\phi_2), \quad (1)$$

where $\phi_1=\phi(-d/2)$ and $\phi_2=\phi(d/2)$. In Eq. (1) $f(\phi,d\phi/dz;z)$ is the bulk energy density containing the terms describing the coupling of the nematic liquid crystal with the distorting field and the elastic contribution deriving from the Frank energy density [1]. The other terms, indicated by $g_1(\phi_1)$ and $g_2(\phi_2)$, describe the surface contributions to the total energy [2]. The first variation of F given by Eq. (1) is [13]

$$\delta F = \int_{-d/2}^{d/2}\left(\frac{\partial f}{\partial \phi} - \frac{d}{dz}\frac{\partial f}{\partial(d\phi/dz)}\right)\delta\phi dz + \left(-\frac{\partial f}{\partial(d\phi/dz)} + \frac{dg_1}{d\phi_1}\right)\delta\phi_1 + \left(\frac{\partial f}{\partial(d\phi/dz)} + \frac{dg_2}{d\phi_2}\right)\delta\phi_2. \quad (2)$$

Since the stable nematic profile has to minimize F, $\delta F=0$ for all $\delta\phi$ belonging to the class of functions continuous with their first derivative (class $\mathcal{C}_1$). It follows that the tilt angle profile $\phi=\phi(z)$ is solution of the bulk equation

$$\frac{\partial f}{\partial \phi} - \frac{d}{dz}\frac{\partial f}{\partial(d\phi/dz)} = 0 \quad (3)$$

and satisfies the boundary conditions

$$-\frac{\partial f}{\partial(d\phi/dz)} + \frac{dg_1}{d\phi_1} = 0,$$

$$\frac{\partial f}{\partial(d\phi/dz)} + \frac{dg_2}{d\phi_2} = 0. \quad (4)$$

Let us indicate by $\phi(z)$ the actual nematic profile, solution of Eq. (3) with the boundary conditions Eq. (4), and $\psi(z)=\phi(z)+\delta\phi(z)$ another profile close to $\phi(z)$. We can modify the actual profile $\phi(z)$ by means of external torques, until to reach the new deformed state $\psi(z)$. The work we have to do by means of the external torques, in a quasistatic process, coincides with δF, given by Eq. (2). It follows that the elastic properties of the nematic material are responsible for a bulk density of torque given by

$$\tau_b^e = -\left(\frac{\partial f}{\partial \phi} - \frac{d}{dz}\frac{\partial f}{\partial(d\phi/dz)}\right), \quad (5)$$

and for a surface density of torque due to the surface treatment and to the elastic deformation given by

$$\tau_{s,1} = -\left(-\frac{\partial f}{\partial(d\phi/dz)} + \frac{dg_1}{d\phi_1}\right),$$

$$\tau_{s,2} = -\left(\frac{\partial f}{\partial(d\phi/dz)} + \frac{dg_2}{d\phi_2}\right) \quad (6)$$

on the surfaces at $z=-d/2$ and $z=d/2$, respectively. By taking into account of this observation, we can conclude that the stable state is the one corresponding to the configuration where the torque densities vanish [13].

To describe the dynamics of the orientation induced by an external field, or the relaxation of an imposed deformation when the distorting field is removed, it is necessary to introduce the dissipation function [14,15], responsible for viscous torques in the bulk, τ_b^v, and at the surfaces, τ_1^v and τ_2^v, at $z=\pm d/2$, respectively. By neglecting inertial effects, the bulk evolution of the nematic deformation is governed by the differential equation $\tau_b^e+\tau_b^v=0$, which has to be solved with the boundary conditions $\tau_{s,1}+\tau_1^v=0$ and $\tau_{s,2}+\tau_2^v=0$. We assume that the bulk viscous torque is

$$\tau_b^v = -\eta_b\frac{\partial \phi}{\partial t}, \quad (7)$$

where η_b is the bulk rotational viscosity [1], and the surface viscous torques are

$$\tau_1^v = -\eta_s\left(\frac{\partial \phi}{\partial t}\right)_1 \quad \text{and} \quad \tau_2^v = -\eta_s\left(\frac{\partial \phi}{\partial t}\right)_2, \quad (8)$$

as suggested by Petrov and Derzhanskii [3,16]. In the following we limit our considerations to the symmetric case where $g_1=g_2$, and hence $\phi(z,t)=\phi(-z,t)$. In this framework the fundamental bulk equation of the dynamical problem under consideration is

$$-\frac{\partial f}{\partial \phi} + \frac{\partial}{\partial z}\frac{\partial f}{\partial(\partial\phi/\partial z)} - \eta_b\frac{\partial \phi}{\partial t} = 0, \quad (9)$$

which has to be solved with the boundary condition

$$-\frac{\partial f}{\partial(\partial\phi/\partial z)} + \frac{dg_1}{d\phi_1} + \eta_s\frac{\partial \phi}{\partial t} = 0, \quad (10)$$

at $z=-d/2$. We are interested in the relaxation of a given distortion, induced on the nematic liquid crystal by an external field, when the external field is removed [11].

III. RELAXATION OF AN IMPOSED DEFORMATION

We consider a nematic liquid crystal, in homeotropic orientation, in the absence of the distorting field. The dielectric anisotropy of the liquid crystal, ε_a, is supposed negative, in such a manner that an electric field parallel to the z axis can deform the initial homeotropic orientation. In the presence of the external field the bulk energy density is [1]

$$f = \frac{1}{2}k\left(\frac{\partial \phi}{\partial z}\right)^2 - \frac{1}{2}\varepsilon_a E^2\cos^2\phi, \quad (11)$$

and the surface energy is assumed of the type (see [2])

$$g_1 = -(w/2)\cos^2\phi_1. \quad (12)$$

When $E=E(t)$ the tilt angle $\phi=\phi(z,t)$ is solution of the partial bulk differential equation

$$k\frac{\partial^2\phi}{\partial z^2} - \frac{1}{2}\varepsilon_a E^2(t)\sin(2\phi) = \eta_b\frac{\partial \phi}{\partial t}, \quad (13)$$

and satisfies the boundary condition

$$-k\frac{\partial \phi}{\partial z} + \frac{1}{2}w\sin(2\phi) + \eta_s\frac{\partial \phi}{\partial t} = 0, \quad (14)$$

at $z=-d/2$, as it follows from Eqs. (9) and (10) and Eq. (11). In order to simplify as much as possible the problem, we assume that the amplitude of the deformation $\phi \ll 1$ in such a manner that Eqs. (13) and (14) can be linearized, and written as

$$k\frac{\partial^2\phi}{\partial z^2}-\varepsilon_a E^2(t)\phi=\eta_b\frac{\partial\phi}{\partial t}, \tag{15}$$

and

$$-k\frac{\partial\phi}{\partial z}+w\phi+\eta_s\frac{\partial\phi}{\partial t}=0, \tag{16}$$

respectively. We consider the case where $E=E(t)$ is $E(t\leq 0)=E_0$ constant, $E(0\leq t\leq t^*)=\mathcal{E}(t)$, with $\mathcal{E}(0)=E_0$ and $\mathcal{E}(t^*)=0$, and $E(t\geq t^*)=0$. Since we are interested in the relaxation of an imposed deformation when the deforming field is removed, we limit our analysis to the situation in which $t^*\to 0$.

We indicate by $\Phi(z)$ the initial deformation in the liquid crystal (for $t\leq 0$), and assume that $\Phi(z)$ belongs to the space of functions of $L_2(-d/2,d/2)$, i.e. whose square is integrable in $(-d/2,d/2)$. It is solution of the ordinary differential equation

$$k\frac{d^2\Phi}{dz^2}-\varepsilon_a E_0^2\Phi=0, \tag{17}$$

with the boundary condition

$$-k\frac{d\Phi}{dz}+w\Phi=0, \tag{18}$$

at $z=-d/2$. When the distorting field is removed, at $t=t^*>0$, the relaxation of the imposed distortion is described by the linear partial differential equation

$$k\frac{\partial^2\phi}{\partial z^2}=\eta_b\frac{\partial\phi}{\partial t}, \tag{19}$$

which has to be solved with the boundary condition

$$-k\frac{\partial\phi}{\partial z}+w\phi+\eta_s\frac{\partial\phi}{\partial t}=0, \tag{20}$$

at $z=-d/2$. We look for a solution of Eq. (19) with boundary condition (20) of the type $\phi(z,t)=Z(z)T(t)$. Simple calculations give

$$\phi(z,t)=\sum_{n=0}^{\infty}C_n\cos(\sigma_n z)\exp(-t/\tau_n), \tag{21}$$

where $\tau_n=(\eta_b/k\sigma_n^2)$, and the wave vectors are the solutions of the equation

$$\left(\sigma_n\frac{d}{2}\right)\left\{\left(\sigma_n\frac{d}{2}\right)\frac{2\eta_s}{\eta_b d}+\tan\left(\sigma_n\frac{d}{2}\right)\right\}=\frac{d}{2b}, \tag{22}$$

where $b=k/w$ is the extrapolation length [1]. Equation (22) has an infinite number of solutions. We shall prove that we can approximate, in $L_2(-d/2,0)$, the initial profile $\phi(z,0)=\Phi(z)$ by means of the functions $\varphi_n(z)=\cos(\sigma_n z)$ which are the eigenfunctions of the operator $\mathcal{A}$ introduced below.

The eigenfunctions $\varphi_n(z)=\cos(\sigma_n z)$ are not orthogonal, but can be orthogonalized by means of the Schmidt technique [17]. The quantity $2\eta_s/(\eta_b d)$ is very small with respect to 1 because the surface viscosity is related to the presence of dissipative effects confined in a mesoscopic region of thickness $\ell\ll d$. Consequently, the first eigenvalue of Eq. (22), is, practically, given by $(\sigma_1 d/2)\tan(\sigma_1 d/2)=d/(2L)$, which coincides with the relation of Rapini-Papoular defining the wave vector of the instability related to the external field in Eq. (17) with boundary condition (18) [2].

The coefficients of the series Eq. (21) are determined by the boundary condition $\phi(z,0)=\Phi(z)$, which in the present case reads

$$\Phi(z)=\sum_{n=0}^{\infty}C_n\cos(\sigma_n z). \tag{23}$$

From Eqs. (19) and (20), by taking into account Eqs. (21) and (23), we get that the function $\Phi(z)$ representing the initial deformation should satisfy the compatibility condition

$$-k\frac{d\Phi}{dz}+w\Phi+k\frac{\eta_s}{\eta_b}\frac{d^2\Phi}{dz^2}=0, \tag{24}$$

that, in general, cannot be satisfied for the presence of the last term [11]. This apparent absurd result follows from the condition that the time derivative of the tilt angle evaluated at the border $z=-d/2$ by means of the bulk Eq. (19) and by means of the boundary condition Eq. (20) have to be equal at $t=0$. However, a simple inspection allows to understand that this result is connected with the hypothesis that the distorting field is removed suddenly, i.e., that $t^*\to 0$. In a real case from Eqs. (13) and (33) it follows that

$$\frac{\partial\phi}{\partial t}=\frac{1}{\eta_b}\left\{k\frac{\partial^2\phi}{\partial z^2}-\frac{1}{2}\varepsilon_a E^2(t)\sin(2\phi)\right\}, \tag{25}$$

in the bulk, and

$$\frac{\partial\phi}{\partial t}=\frac{1}{\eta_s}\left\{k\frac{\partial\phi}{\partial z}+\frac{1}{2}w\sin(2\phi)\right\}, \tag{26}$$

at $z=-d/2$. At $t=0$ from Eqs. (25) and (26), taking into account that $E(0)=E_0$, and Eqs. (17) and (18), it follows that the initial time derivative of the tilt angle is zero in all $-d/2\leq z\leq d/2$, up to the border $z=-d/2$, without any problem of compatibility connected with the bulk differential equation or with the boundary condition. In other words, the compatibility condition (24) is related to the assumption that Eq. (19) is valid up to $t=0$, not to the presence of the surface viscosity in the boundary condition [11]. The analysis of the relaxation of an imposed deformation in terms of Eq. (19) with the boundary condition Eq. (20) is then valid only for $t\geq t^*$. In this case there are no problems of compatibility between the time derivatives of the tilt angle evaluated at the surface by means of the bulk equation and of the boundary condition, if the solution of the dynamical problem is obtained as described above.

For $t\geq t^*$ from the bulk Eq. (19) and from the boundary condition Eq. (20) we get

$$\left(\frac{\partial\phi}{\partial t}\right)_{z=-d/2}=\frac{k}{\eta_b}\left(\frac{\partial^2\phi}{\partial z^2}\right)_{z=-d/2}, \tag{27}$$

$$\left(\frac{\partial \phi}{\partial t}\right)_{z=-d/2} = \frac{1}{\eta_s}\left(k\frac{\partial \phi}{\partial z} - w\phi\right)_{z=-d/2}, \quad (28)$$

respectively. By substituting expansion (21) into Eqs. (27) and (28) we obtain

$$\left(\frac{\partial \phi}{\partial t}\right)_{z=-d/2} = \frac{k}{\eta_b}\sum_{n=0}^{\infty} \sigma_n^2 C_n \cos\left(\sigma_n \frac{d}{2}\right)\exp(-t/\tau_n), \quad (29)$$

$$\left(\frac{\partial \phi}{\partial t}\right)_{z=-d/2} = \frac{1}{\eta_s}\sum_{n=0}^{\infty}\left\{k\sigma_n \sin\left(\sigma_n \frac{d}{2}\right) + w\cos\left(\sigma_n\frac{d}{2}\right)\right\}C_n \exp(-t/\tau_n). \quad (30)$$

The expressions for the time derivative at the border evaluated by means of Eqs. (29) and (30) are identical when the wave vectors σ_n are solutions of the eigenvalues Eq. (21). As expected, for $t \geq t^*$, the time derivatives of the nematic tilt angle at the border evaluated by means of the bulk differential equation and by means of the boundary condition coincide.

IV. WELL POSEDNESS OF THE PROBLEM

Although it is possible to present several mathematical descriptions for a field $E(t)$ which describes the fast transient, we cannot expect that such functions have a real physical meaning. So, we take a different route, which disregards the time interval $[0,t^*]$ and the unknown function $E(t)$. This route is suggested by the standard treatment of the heat conduction problem, in dimensionless units,

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial \zeta^2}, \quad 0 < \zeta < \pi, \quad \theta(t,0) = 0, \quad \theta(t,\pi) = 0, \quad (31)$$

where $\theta(\zeta,t)$ is the temperature in ζ at the time t. Let the temperature for $t \leq 0$ be identically 1. Then, the accepted solution of this problem is

$$\theta(t,\zeta) = \frac{4}{\pi}\sum_{n=1}^{+\infty}\frac{\sin(n\zeta)}{n}e^{-n^2 t}. \quad (32)$$

We note however that $\theta(t,0) = \theta(t,\pi) = 0$ for $t \geq 0$ and the temperature cannot change abruptly. So, there should be a time interval $[0,t^*]$ in which the problem has the form

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial \zeta^2} + \mathcal{G}(t,\zeta;\theta),$$

$$\theta(t,0) = h_1(t), \quad \theta(t,\pi) = h_2(t),$$

$$h_1(0) = h_2(0) = 1, \quad h_1(t^*) = h_2(t^*) = 0,$$

where $\mathcal{G}(t,\zeta;\theta)$ depends on the system under consideration and how the external temperature has been changed.

It is a fact that nobody care about this time interval, thanks to the following properties of series (32): (i) the series and its derivatives converge uniformly on every strip $[\tau,+\infty) \times [0,\pi]$, for every $\tau > 0$ and when the series is replaced in the heat equation equality holds in this strip. (ii) The initial condition $\lim_{t\to 0+}\theta(t,x) = 1$ is not satisfied in the sense of uniform or pointwise convergence on the interval $[0,\pi]$. But, the following condition holds:

$$\lim_{t\to 0+}\int_0^{\pi} |\theta(t,\zeta) - 1|^2 d\zeta = 0. \quad (33)$$

The transient on the short time interval $[0,t^*]$ can be safely ignored thanks to these properties which are consequence of the fact that $\{\sin(nx)\}$ is a complete orthonormal system in $L^2(0,\pi)$ and of the fast decaying factors $e^{-n^2 t}$, $n \geq 1$. Thanks to the fact that the heat equation enjoys the previous properties, we say that it describes a well posed problem.

We are going to prove that similar properties hold also for the problem [Eqs. (19) and (20)] so that, precisely as in the case of heat equation, we can disregard the transient field $E(t)$. In order to fulfill this program, we introduce the Hilbert space $X = \mathbf{R} \times L^2(-d/2,0)$ with inner product computed componentwise from the following formula

$$\left\langle \begin{bmatrix} r \\ \phi(\cdot) \end{bmatrix}, \begin{bmatrix} \rho \\ \psi(\cdot) \end{bmatrix} \right\rangle = r\rho + \frac{\eta_b}{\eta_s}\int_{-d/2}^{0} \phi(z)\psi(z)dz. \quad (34)$$

Then we define the following operator $\mathcal{A}$:

$$\mathrm{dom}\,\mathcal{A} = \left\{ \begin{bmatrix} r \\ \phi \end{bmatrix}, \quad \phi \in H^2(-d/2,0), \quad \frac{d\phi(0)}{dz} = 0, \quad r = \phi(-d/2) \right\},$$

$$\mathcal{A}\begin{bmatrix} r \\ \phi \end{bmatrix} = \begin{bmatrix} (k/\eta_s)\dfrac{d\phi(-d/2)}{dz} - (w/\eta_s)r \\ (k/\eta_b)\dfrac{d^2\phi(z)}{dz^2} \end{bmatrix}.$$

We note the following:

(i) The elements of class $H^2(-d/2,0)$ are functions which are continuous with continuous first derivative in the closed interval $[-d/2,0]$ and whose second derivative is square integrable on this interval. The second derivative might not exist in a set of (Lebesgue) measure zero. Continuity of $\phi(z)$ and $\phi'(z)$ in the closed interval $[-d/2,0]$ shows that the conditions at $z=0$ and $z=-d/2$ can be imposed.

(ii) The condition $(d\phi/dz)(0)=0$ has been introduced since in fact the domain is $(-d/2,+d/2)$ but the functions of interest are even functions, in our symmetric problem. So we confine ourselves to study the problem on $(-d/2,0)$ and we impose the condition $\phi'(0)=0$. The definition of $\mathcal{A}$ is suggested by Eqs. (19) and (20). The computation of its eigenvalues leads to Eq. (22).

(iii) The Hilbert space $X=\mathbf{R}\times L^2(-h,0)$ is used for example in the study of systems with (finite) memory. In that contest it is denoted M^2. We use the weight η_b/η_s in front of the integral since with this inner product the operator $\mathcal{A}$ is self-adjoint.

(iv) Partial differential equations with "dynamical" boundary conditions have been studied initially in problems of probability theory (systems with "Wentzel boundary conditions"); see [18,19]. Recently they have been encountered in the study of phase transition and Cahn-Hillard equations (see [20]). Systems with dynamical boundary conditions are currently studied in product spaces of the type $X=\mathbf{R}\times L^2(-d/2,0)$, in the framework of (holomorphic) semigroup theory; and the operator $\mathcal{A}$ above is the infinitesimal generator of the semigroup. So, we might deduce part of the results we need from existing abstract results. Instead, we present a spectral analysis of the operator $\mathcal{A}$, which is different, since we shall see that it leads to important conclusions. As we noted (and as can be easily checked) the operator $\mathcal{A}$ is self-adjoint. So, its eigenvalues are real [for this reason we confined ourselves to the real Hilbert space $\mathbf{R}\times L^2(-d/2,0)$].

Now we compute the inverse $\mathcal{A}^{-1}$. A simple computation shows that

$$\mathcal{A}^{-1}\begin{bmatrix}\rho\\ \psi\end{bmatrix}=\begin{bmatrix}-\dfrac{\eta_b}{w}\displaystyle\int_{-d/2}^{0}\psi(s)ds-\dfrac{\eta_s}{w}\rho\\ -\dfrac{\eta_s}{w}\rho-\dfrac{\eta_b}{k}\displaystyle\int_{-d/2}^{0}\left(\dfrac{d}{2}-\dfrac{k}{w}+s\right)\psi(s)ds+\dfrac{\eta_b}{k}\displaystyle\int_{0}^{z}(z-s)\psi(s)ds\end{bmatrix}.$$

This shows not only that $\mathcal{A}^{-1}$ exists, but also that it is a compact operator. So, the theory of self-adjoint operator with compact resolvent shows that the sequence of the normalized eigenvectors is an orthonormal basis of $X=\mathbf{R}\times L^2(-d/2,0)$.

Note that every eigenvector is an element of X, so it has a first component, let us call it $r\in\mathbf{R}$ and a second component, let it be $\phi(z)\in L^2(-d/2,0)$. But, the eigenvector must belong to the domain of $\mathcal{A}$ so that it must be $r=\varphi(-d/2)$.

It is convenient to introduce the following notations: (1) let $\{h_n\}$ and $\{k_n\}$ be two sequences (in general, in two different Hilbert spaces). We define $h_n\sim k_n$ when there exist $m>0$ and M such that for every n the following hold: $m\|k_n\|\le\|h_n\|\le M\|k_n\|$. (2) Let $\{\alpha_n\}$ be a sequence of numbers. The notations $\{\alpha_n\}\in l^2$ mean that $\Sigma|\alpha_n|^2<+\infty$.

The computation of the eigenvalues/eigenvectors of the operator $\mathcal{A}$ leads to the problem of finding nonzero solutions of

$$\begin{cases}\dfrac{k}{\eta_b}\dfrac{d^2\phi(z)}{dz^2}=\lambda\phi(z),\\ \lambda r=\dfrac{k}{\eta_s}\dfrac{d\phi(-d/2)}{dz}-\dfrac{w}{\eta_s}r\end{cases}\qquad \phi'(0)=0,\quad r=\phi(-d/2).$$

A nonzero solution exists if and only if $\lambda=-(k/\eta_b)\sigma^2$ where $\sigma=\sigma_n$ solves Eq. (22). Note that we get the same eigenvalue for σ and $-\sigma$.

An eigenvector which corresponds to $\lambda_n=-k\sigma_n^2/\eta_b$, $n\ge0$, is

$$\Theta_n=\begin{bmatrix}\cos(\sigma_n d/2)\\ \cos(\sigma_n z)\end{bmatrix},$$

$$\|\Theta_n\|^2=[\cos(\sigma_n d/2)]^2+\frac{\eta_b}{2\eta_s}\left[\frac{d}{2}+\frac{1}{2\sigma_n}\sin(\sigma_n d)\right].$$

This sequence of eigenvector is not normalized, i.e., the norm is not equal to 1, but we shall see that it is *almost normalized*, i.e. that there exists m and M such that

$$0<m<\|\Theta_n\|<M. \tag{35}$$

The projection of the eigenvectors on $L^2(-d/2,0)$ spans this space. This justifies the fact that every function $f(z)\in L^2(-d/2,0)$ can be approximated by linear combinations of the functions $\varphi_n(z)=\cos\sigma_n z$. But these functions need not be a basis of $L^2(-d/2,0)$. In order to clarify this point, and to prove Eq. (35), we need the following asymptotic estimate, obtained in the Appendix:

$$\sigma_n=\frac{2}{d}\left\{(n\pi-\pi/2)+\frac{d\eta_b}{(2\eta_s\pi)n}+o(1/n).\right\} \tag{36}$$

Using these estimates, we see that

$$[\cos(\sigma_n d/2)]^2=\left[\frac{d\eta_b}{(2\eta_s\pi)n}+o(1/n)\right]^2\sim\frac{1}{n^2} \tag{37}$$

so that we have the following estimate, which implies Eq. (35),

$$\|\Theta_n\|^2\sim\frac{d\eta_b}{4\eta_s}. \tag{38}$$

We said already that every element of $X=\mathbf{R}\times L^2(-d/2,0)$ can be expanded in series of Θ_n. Every element of $X=\mathbf{R}\times L^2(-d/2,0)$ can be represented as a sum

$$\begin{bmatrix} c \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ f(x) \end{bmatrix}.$$

In particular, we have the representation

$$\begin{bmatrix} c \\ 0 \end{bmatrix} = \sum_{n=0}^{+\infty} \left\langle \begin{bmatrix} c \\ 0 \end{bmatrix}, \frac{\Theta_n}{\|\Theta_n\|} \right\rangle \frac{\Theta_n}{\|\Theta_n\|}$$
$$= \sum_{n=0}^{+\infty} \frac{1}{\|\Theta_n\|^2}[c\cos(\sigma_n d/2)] \begin{bmatrix} \cos(\sigma_n d/2) \\ \cos(\sigma_n z) \end{bmatrix}. \quad (39)$$

Note that

$$\left\{ \sum_n \frac{1}{\|\Theta_n\|_n^2}[c\cos(\sigma_n d/2)] \right\} < +\infty$$

thanks to estimates (37) and (38).

Remark 1. For further use we note that this representation *is unique* and the coefficient of $\Theta_0/\|\Theta_0\|$ is

$$c\frac{\cos(\sigma_0 d/2)}{\|\Theta_0\|}.$$

If $c\neq 0$ then this coefficient is different from zero, since $\sigma_0 d/2 \neq \pi/2$.

The bottom row in Eq. (39) shows that

$$\sum_{n=0}^{+\infty} \frac{1}{\|\Theta_n\|^2}[c\cos(\sigma_n d/2)]\cos\sigma_n z = 0$$

so that the sequence $\{\cos(\sigma_n z)\}_{n\geq 0}$ can be used to approximate every $f(z)\in L^2(-d/2,0)$ but it is not linearly independent, hence it is not a basis.

We are going to prove that we have a basis by removing the first element $\cos\sigma_0 z$. Namely, we prove in the Appendix that the sequence $\{\varphi_n(z)=\cos(\sigma_n z)\}_{n\geq 1}$ is a Riesz basis in $L^2(-d/2,0)$. A Riesz basis is a sequence of functions which can be transformed to an orthonormal basis using a bounded and boundedly invertible transformation (i.e., a coordinate transformation).

We draw the following conclusions from the previous arguments: Fix any square integrable initial condition $\phi_0(z)$ for Eq. (19) and any initial condition r_0 for Eq. (20). Do not care about the compatibility condition. By the way, $\phi_0(z)\in L^2(-d/2,0)$ and to compute its value at $-d/2$ is meaningless. The system has the solution

$$\Theta(t) = \begin{bmatrix} r(t) \\ \phi(z,t) \end{bmatrix} = \sum_{n=0}^{+\infty} \alpha_n e^{-(k/\eta_b)\sigma_n^2 t} \begin{bmatrix} \cos(\sigma_n d/2) \\ \cos(\sigma_n z) \end{bmatrix},$$

$$\alpha_n = \left\langle \begin{bmatrix} r_0 \\ \phi_0(z) \end{bmatrix}, \frac{\Theta_n}{\|\Theta_n\|} \right\rangle.$$

Then we have the following:

(i) the function $t\to\Theta(t)$ is continuous from $[0,+\infty)$ to X.

(ii) Let

$$\left\{ \begin{bmatrix} r_n \\ \phi_n \end{bmatrix} \right\} \in \operatorname{dom}\mathcal{A}, \quad \begin{bmatrix} r_n \\ \phi_n \end{bmatrix} \to \begin{bmatrix} \hat{r} \\ \hat{\phi} \end{bmatrix}.$$

Let Θ_n and $\hat{\Theta}$ be the corresponding solutions. Then, $\lim_{n\to+\infty}\Theta_n(t)=\hat{\Theta}(t)$ on $[0,+\infty)$. The limit has to be computed in the norm of X. Note that when an initial condition belongs to dom $\mathcal{A}$ then the compatibility condition $r_n=\phi_n(-d/2)$ is satisfied.

(iii) The series

$$\sum_{n=0}^{+\infty} \alpha_n e^{-(k/\eta_b)\sigma_n^2 t}\cos(\sigma_n z) = \phi(z,t)$$

converges uniformly on $[t^*,+\infty)\times(-d/2,0)$ *for every* $t^*>0$. In particular it is continuous on $[t^*,+\infty)\times(-d/2,0)$ and its value for $z=-d/2$ is $r(t)$; i.e., the compatibility condition is automatically satisfied for every $t>0$, even if it has not been imposed at $t=0$.

These considerations prove that system (19) and (20) has the same properties as the heat equation and this is the result we wanted to achieve: system (19) and (20) is well posed in the same sense as the heat equation. Note that these results have been proved using the weighted inner product in X but different (constant) weights gives rise to the same convergence properties. So, these results hold also if we use the weight equal to 1.

Now we clarify a last problem. We noted that the sequence $\{\cos\sigma_n z\}_{n\geq 0}$ is complete but not independent on $L^2(-d/2,0)$ while $\{\cos(\sigma_n z)\}_{n\geq 1}$ is a Riesz basis. So, we can also represent the solution $\phi(z,t)$ as

$$\sum_{n=1}^{+\infty} \beta_n e^{-(k/\eta_b)\sigma_n^2 t}\cos(\sigma_n z)$$

with coefficients β_n which may be different from the coefficients α_n. The coefficients β_n and α_n coincide if r_0 is so chosen that $\alpha_0=0$, i.e., such that

$$\left\langle \begin{bmatrix} r_0 \\ \phi_0(z) \end{bmatrix}, \frac{\Theta_0}{\|\Theta_0\|} \right\rangle = 0.$$

Finally, the fact that the sequence $\{\cos\sigma_n z\}_{n\geq 0}$ is not independent has no effect on the use of Gram-Schmidt method since any finite section of this sequence, $\{\cos\sigma_n z\}_{0\leq n\leq N}$, is linearly independent.

V. CONCLUSION

From the analysis reported above it follows that a description of the relaxation of a given distortion in nematic liquid crystals based on the concept of bulk and surface viscosity is possible. The apparent compatibility condition that has to satisfy the initial deformation is related to the assumption that the distorting field is removed discontinuously, not to the present of the surface viscosity in the boundary condition.

ACKNOWLEDGMENT

Many thanks are due to E. Virga for useful discussions and for a critical reading of the paper.

APPENDIX

We first prove the asymptotic estimate (36). We proceed as follows: We replace $x=\sigma d/2$ and we see that Eq. (22) takes the form

$$\tan x = -Ax + \frac{B}{x}, \quad A = \frac{2\eta_s}{d\eta_b} > 0, \quad B = \frac{dw}{2k} > 0. \quad (A1)$$

Both the sides of Eq. (A1) are odd functions of x so that the solutions are pairwise opposite in sign. But we noted that $\pm\sigma$, i.e., $\pm x$ leads to the same eigenvalue. So, we can consider the positive roots.

The right-hand side of Eq. (A1) is a decreasing function of x and this shows the existence of precisely one solution x_n in each one of the intervals $(n\pi-\pi/2, n\pi+\pi/2)$, $n\geq 0$. Clearly,

$$\lim_{n\to+\infty} \{(n\pi - \pi/2) - x_n\} = 0$$

since $(n\pi - x_n) \in (-\pi/2, \pi/2)$. If it does not converges to zero then a subsequence $n_k\pi - x_{n_k}$ has to converge to $s\neq 0$. Periodicity of $\tan x$ shows that the left-hand side of Eq. (A1) converges to $\tan(s-\pi/2)$ while the right-hand side diverges. So, we can represent

$$x_n = n\pi - \frac{\pi}{2} + y_n, \quad y_n \to 0$$

and we prove

$$y_n \sim \frac{1}{n}.$$

In fact we have

$$\frac{\sin(y_n - \pi/2)}{\cos(y_n - \pi/2)} = \tan(y_n - \pi/2) = -A[y_n + (n\pi - \pi/2)] + B\frac{1}{y_n + (n\pi - \pi/2)}.$$

This shows that

$$\frac{1}{n}\tan(y_n - \pi/2) \to -A\pi.$$

We know that $y_n + \pi/2 \to \pi/2$ so that we must have $n\cos(y_n - \pi/2) \to 1/(-A\pi)$. This is only possible if

$$y_n \sim \frac{1}{A\pi n}.$$

So, we find

$$x_n = (n\pi - \pi/2) + \frac{1}{A\pi n} + o(1/n)$$

$$= (n\pi - \pi/2) + \frac{d\eta_b}{(2\eta_s\pi)n} + o(1/n), \quad \sigma_n = \frac{2}{d}x_n$$

as wanted.

Now we prove the statement concerning the Riesz basis. We use Bari theorem (see [12]) for this proof. Bari theorem states that every sequence which is "ω independent" and "quadratically close" to an orthogonal basis whose elements have constant norm is a Riesz system.

The sequence $\{\varphi_n\}$ is quadratically close to an orthogonal basis $\{e_n\}$ when

$$\sum_n \|\varphi_n - e_n\|^2 < +\infty.$$

In our case, $\varphi_n(z)=\cos(\sigma_n z)$ and we choose

$$e_n = e_n(z) = \cos\left\{\frac{2}{d}(n\pi - \pi/2)z\right\}.$$

This sequence is an orthogonal basis of $L^2(-d/2,0)$ since it is the sequence of the eigenvectors of the problem

$$\frac{d^2\phi(z)}{dz^2} = \lambda\phi(z), \quad \frac{d\phi(0)}{dz} = 0, \quad \phi(-d/2) = 0.$$

Its elements have norm $\sqrt{d}/2$.

Estimate (36) shows that

$$\left|\cos(\sigma_n z) - \cos\left\{\frac{2}{d}(n\pi - \pi/2)z\right\}\right| \leq \frac{M}{n}. \quad (A2)$$

The constant M does not depend on n. It follows that the sequence $\{\cos(\sigma_n z)\}_{n\geq 1}$ is quadratically close to the sequence $\{\cos[\frac{2}{d}(n\pi - \pi/2)z]\}_{n\geq 1}$.

So, we have to prove "ω independence." This is the property that the following conditions:

$$\{\alpha_n\} \in l^2, \quad \sum_{n=1}^{+\infty} \alpha_n \cos(\sigma_n z) = 0 \quad (A3)$$

hold together only if $\alpha_n=0$. This we prove now.

Observe that series (A3) converges in $L^2(-d/2,0)$. We do not know whether it converges pointwise for some choice of z. But, we know that it converges for $z=-d/2$ thanks to inequality (A3) which for $z=-d/2$ gives $|\cos(\sigma_n d/2)| < M/n$. So,

$$\sum_{n=1}^{+\infty} \alpha_n \cos(\sigma_n d/2) = c \in \mathbf{R}.$$

We have

$$\begin{bmatrix} c \\ 0 \end{bmatrix} = \sum_{n=1}^{+\infty} \alpha_n \begin{bmatrix} \cos(\sigma_n d/2) \\ \cos(\sigma_n z) \end{bmatrix} = \sum_{n=1}^{+\infty} \alpha_n \Theta_n.$$

Note that the index $n=0$ does not appear in the series, i.e. $\alpha_0=0$. Remark 1 implies that $c=0$. But then we have $\alpha_n=0$ for every n since $\{\Theta_n\}$ is an orthogonal sequence. This ends the proof.

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